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STABILITY ANALYSIS ON A DELAY *SIR* MODEL WITH DENSITY DEPENDENT BIRTH RATE

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1. Introduction

The spread process of infectious diseases to a population is often described mathematically by using compartment models. Let us divide the whole population into three components denoted by S, I and R. The $S(t)$ denotes the number of the members of the population who are susceptible to the disease and $I(t)$ is the number of infective members of the population at the present time t. The third component $R(t)$ represents the number of members who have been removed from the possibility of infection through full immunity. The total number of the population is denoted by $N(t) = S(t) + I(t) + R(t)$.

In this paper, we shall analyze the stability property of a delayed *SIR* disease transmission model with density dependent birth rate. The model is described as follows:

$$\frac{d}{dt}S(t) = -\beta S(t)\int_0^h f(s)I(t-s)ds - \mu_1 S(t) + bN(t)$$

$$\frac{d}{dt}I(t) = \beta S(t)\int_0^h f(s)I(t-s)ds - (\mu_2+\lambda)I(t) \tag{1}$$

$$\frac{d}{dt}R(t) = \lambda I(t) - \mu_3 R(t),$$

where h, β, b, λ, μ_1, μ_2 and μ_3 are positive constants and $f(s)$ is a nonnegative and continuous function on $[0,h]$. In order not to change the values of corresponding equilibrium points between (1) and the system without delay effects, we assume that

$$\int_0^h f(s)ds = 1.$$

Model (1) describes infectious process of the disease transmitted by vectors (see [3, 4, 5]). It is natural from the biological point of view to assume that when a susceptible vector is infected by an infected person, there is a time during which the infectious agents develop in the vector and it is only after that time that the infected vector itself becomes infectious. Hence, the integral term in (1)

$$\beta S(t)\int_0^h f(s)I(t-s)ds$$

involves the delay effect in the disease transmission process. The transmission of infection is expressed by law of mass-action. The $f(s)$ is the fraction of vector population in which the time taken to become infectious is s, which satisfies that $0 \le s \le h$. It may be realistic to assume that the time has some upper bound h, which is a finite number. The β is the average number of contacts per infective per day.

Further, the μ_1, μ_2 and μ_3 express death rates of the susceptibles, infectives and recovered, respectively. Since the epidemic will increase the death rates of the infectives and recovered (or at least the rate of infectives), it may be natural biologically to assume that

$$\mu_1 \le \min\{\mu_2,\ \mu_3\}.$$

The λ represents the recovery rate of the infectives and b is the birth rate constant of the population. The model (1) assumes that the birth process is density dependent and the growth of the number of newborns (who are assumed to enter into the susceptible class, that is, we do not consider the possibility of the vertical transmission of the disease) is proportional to the total number of the population $N(t)$.

If we ignore both the effect of time delays for the disease transmission process and the density dependence in the birth process (that is, if we replace in model (1) $\beta S(t)\int_0^h f(s)I(t-s)ds$ and $bN(t)$ with $\beta S(t)I(t)$ and b, respectively) and further assume that the birth rate and all death rates are identical ($\mu_1=\mu_2=\mu_3=b$), then we have a system of ordinary differential equations, which was considered by Hethcote [7]. Clearly, the system satisfies that $N(t)\to 1$ as $t\to\infty$ and can be reduced to the plane system. Hethcote [7] showed that the disease free equilibrium point (where only the susceptible class persists, and the infective and recovered classes become extinct) is globally asymptotically stable if the endemic equilibrium point (where all three classes persist) does not exist. Further, the endemic point is proved to be globally asymptotically stable whenever it exists (see also [1]).

For the system with the delayed disease transmission process and with different b and μ_i ($i=1,2,3$), but without a density dependent birth process (that is, for system (1) with b instead of $bN(t)$), Takeuchi, Ma and Beretta [9] considered the effect of delay on the asymptotic stability of the disease free or endemic equilibrium points and proved the following:

(i) the disease free equilibrium point is globally asymptotically stable if the endemic equilibrium does not exist;

(ii) the endemic equilibrium is locally asymptotically stable if it exists;

(iii) if there is some $\tilde{S}$ satisfying $S^*<\tilde{S}<b/(\mu_2+\lambda)$ such that the following two conditions hold true

$$h<\min\{(2\beta\tilde{S})^{-1},(\tilde{S}-S^*)/(b-\mu_1S^*)\};$$

$$b<\tilde{S}\{\beta[b/(\mu_2+\lambda)-\tilde{S}]+\mu_1\},$$

where S^* is the number of the susceptibles at the endemic point, then the endemic equilibrium is globally asymptotically stable.

These results show that delay is harmless on global asymptotic stability of the disease free equilibrium point and also on local stability of the endemic equilibrium point.

In this paper we consider system (1) with a density ***dependent*** birth process, whose dynamical behavior is qualitatively different from that of the system with a density ***independent*** birth process. For the system with density independent process, the endemic equilibrium point is always locally asymptotically stable if it exists and can be globally asymptotically stable under the effect of small delay [9]. But for system (1), the endemic equilibrium point can be unstable when $h = \infty$ (see Section 4).

The initial condition of (1) is given as

$$S(t_0 + s) = \varphi_1, \; I(t_0 + s) = \varphi_2, \; R(t_0 + s) = \varphi_3, \quad -h \leq s \leq 0, \qquad (2)$$

where $t_0 \geq 0$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$ such that $\varphi_i \geq 0$ and $\varphi_i(0) > 0$ for $i = 1, 2, 3$. The C denotes the Banach space $C([-h, 0], R^3)$ of continuous functions mapping the interval $[-h, 0]$ into R^3.

It is easy to check that the solution $(S(t), I(t), R(t))^T$ of (1) satisfying the initial condition (2) exists and is unique for all $t \geq t_0$ (see [6] or [8]). Also it is trivial that the solution is positive, that is, $S(t) > 0, I(t) > 0$ and $R(t) > 0$ for all $t \geq t_0$.

Let us consider the nonnegative equilibrium points of system (1).

System (1) always has a trivial equilibrium point

$$E_0 = (0, \; 0, \; 0)$$

which exhibits extinction of the population.

If $b = \mu_1$, then for any $s > 0$,

$$E_s = (s, \; 0, \; 0)$$

is a boundary equilibrium point (the disease free equilibrium point) of (1).

If

$$\mu_1 < b < \mu_3(\mu_2 + \lambda)/(\mu_3 + \lambda), \tag{3}$$

then system (1) also has a positive equilibrium point (the endemic equilibrium point)

$$E_+ = (S^*,\ I^*,\ R^*),$$

where

$$S^* \equiv \frac{\mu_2 + \lambda}{\beta}, \quad I^* \equiv \frac{\mu_3(b - \mu_1)S^*}{\beta S^* \mu_3 - b(\mu_3 + \lambda)}, \quad R^* \equiv \frac{\lambda}{\mu_3} I^*.$$

Note that $\mu_3(\mu_2 + \lambda)/(\mu_3 + \lambda) > \mu_1$ because of the assumption that $\mu_1 \leq \min\{\mu_2,\ \mu_3\}$.

2. Stability analysis on E_0 and E_s

This section considers the asymptotic behavior of the solution of (1) for the case where the endemic equilibrium point E_+ does not exist, that is, the case where $b \leq \mu_1$ or $b \geq \mu_3(\mu_2 + \lambda)/(\mu_3 + \lambda)$.

First we consider stability of E_0.

Theorem 1. (a) *If $\mu_1 > b$, then E_0 is globally asymptotically stable.*

(b) *If $b > \mu_1$, then E_0 is unstable.*

(c) *Further, if $b > \mu_3(\mu_2 + \lambda)/(\mu_3 + \lambda)$, then $N(t) = S(t) + I(t) + R(t) \to +\infty$ as $t \to \infty$.*

Proof. Conclusion (a) is obvious by the following inequality

$$\frac{d}{dt}(S(t) + I(t) + R(t)) = \frac{d}{dt}N(t) \leq -(\mu_1 - b)N(t)$$

for all $t \geq t_0$.

Note that the linearized system of (1) at E_0 is

$$\begin{aligned}
\frac{d}{dt}S(t) &= (b - \mu_1)S(t) + bI(t) + bR(t) \\
\frac{d}{dt}I(t) &= -(\mu_2 + \lambda)I(t) \\
\frac{d}{dt}R(t) &= \lambda I(t) - \mu_3 R(t).
\end{aligned}$$

We see that E_0 is unstable if $b > \mu_1$.

Now let us consider the case (c). It is possible to choose a positive constant ε such that $(\mu_2+\lambda-b)/\lambda < \varepsilon < b/\mu_3$ by the assumption. Then, from (1) we have that for $t \geq t_0$,

$$\begin{aligned}\frac{d}{dt}(S(t)+I(t)+\varepsilon R(t)) &= (b-\mu_1)S(t)+(b-\mu_2-\lambda+\varepsilon\lambda)I(t) \\ &\quad +(b-\varepsilon\mu_3)R(t) \\ &\geq \delta(S(t)+I(t)+\varepsilon R(t)),\end{aligned}$$

where

$$\delta = \min\{b-\mu_1, b-\mu_2-\lambda+\varepsilon\lambda, (b-\varepsilon\mu_3)/\varepsilon\} > 0$$

by the definition of ε. Thus,

$$S(t)+I(t)+\varepsilon R(t) \to +\infty \text{ as } t \to \infty,$$

from which we see that

$$S(t)+I(t)+R(t) \to +\infty \text{ as } t \to \infty.$$

This proves Theorem 1.

Next, let us consider the remaining case where no endemic equilibrium point exist.

Theorem 2. *If $\mu_1 = b$, then for any solution $(S(t), I(t), R(t))^T$ of* (1) , *there is some constant $c \geq 0$ such that $c \leq S^* = (\mu_2+\lambda)/\beta$ and*

$$\lim_{t\to+\infty} S(t) = c, \quad \lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} R(t) = 0.$$

Proof. Set

$$G = \{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C \mid \varphi_1 \geq 0, \varphi_2 \geq 0, \varphi_3 \geq 0\}.$$

Clearly, G is invariant for (1). Moreover, we can easily show that the solutions of (1) are bounded when $\mu_1 = b$. For $\varphi \in G$, let us define the following Liapunov function

$$V(\varphi) = \varphi_1(0) + \omega_1\varphi_2(0) + \omega_2\varphi_3(0) + \omega_3(\varphi_1(0)+\varphi_2(0)),$$

where ω_1, ω_2 and ω_3 are some positive constants chosen later. Then, the time derivative of $V(\varphi)$ along the solutions of (1) is

$$\begin{aligned} \dot{V}(\varphi)|_{(1)} &= -(1-\omega_1)\beta\varphi_1(0)\int_0^h f(s)\varphi_2(-s)ds \\ &\quad -\left[\omega_1(\mu_2+\lambda)+\omega_3(\mu_2+\lambda-b)-\omega_2\lambda-\mu_1\right]\varphi_2(0) \\ &\quad -\left[\omega_2\mu_3-\omega_3\mu_1-\mu_1\right]\varphi_3(0). \end{aligned}$$

Here we used the condition $\mu_1 = b$. It is possible to choose $\omega_i > 0$ $(i = 1,2,3)$ such that

$$\omega_1 < 1,$$

$$\omega_1(\mu_2+\lambda)+\omega_3(\mu_2+\lambda-b)-\omega_2\lambda-\mu_1 > 0$$

and

$$\omega_2\mu_3-\omega_3\mu_1-\mu_1 > 0,$$

because of $\mu_1 = b \le \min\{\mu_2,\ \mu_3\}$. Thus, $V(\varphi)$ is a Liapunov function on the subset G in C. Let

$$Q = \left\{\varphi \in G \mid \dot{V}(\varphi)|_{(1)} = 0\right\}.$$

Then, $\dot{V}(\varphi) = 0$ if and only if $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = 0$ or $\varphi_3(0) = \varphi_2 = 0$. If $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = 0$, then $\varphi_1 = \varphi_2 = \varphi_3 = 0$ by (1). If $\varphi_3(0) = \varphi_2 = 0$, then, again by (1) and $\mu_1 = b$, we have that $\varphi_3 = 0$ and $\dot{\varphi}_1(0) = 0$, which implies that $\varphi_1 \equiv c \ge 0$ for some constant c. Therefore, by the Liapunov-LaSalle invariance principle for functional differential equations (see, for example, [8]) we have that

$$\lim_{t\to+\infty} S(t) = c, \quad \lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} R(t) = 0.$$

Now let us further show that $c \le S^* = (\mu_2+\lambda)/\beta$, which actually gives an eventual upper bound on $S(t)$.

In fact, if $c > (\mu_2+\lambda)/\beta$ (hence, $c \ne 0$), then for sufficiently small $\varepsilon > 0$, there is a sufficiently large $\bar{t} > t_0$ such that $S(t) \ge c-\varepsilon > 0$ and $\beta(c-\varepsilon)-(\mu_2+\lambda) > 0$ for $t \ge \bar{t}$. Thus, from (1) we have that for $t \ge \bar{t}$,

$$\frac{d}{dt}I(t) \ge \beta(c-\varepsilon)\int_0^h f(s)I(t-s)ds - (\mu_2+\lambda)I(t). \tag{4}$$

Define

$$W(t) = I(t) + \beta(c-\varepsilon)\int_0^h f(s)\int_{t-s}^t I(u)duds.$$

Then, it is easy to see that for $t \geq t_0$, $W(t) > 0$ and $\lim_{t\to+\infty} W(t) = 0$, since $\lim_{t\to+\infty} I(t) = 0$ and h is finite.

On the other hand, from (4) we have that the time derivative of $W(t)$ along the solutions of (1) for $t \geq \bar{t}$ becomes

$$\dot{W}(t)|_{(1)} \geq (\beta(c-\varepsilon) - (\mu_2 + \lambda))\, I(t) > 0,$$

which clearly implies that $\lim_{t\to+\infty} W(t) > 0$. This is a contradiction to that $\lim_{t\to+\infty} W(t) = 0$. This proves that $c \leq S^* = (\mu_2+\lambda)/\beta$. The proof of Theorem 2 is completed.

3. Convergence on E_+

In the following, we assume (3), that is, that there exists E_+ and consider its stability property.

By changing the variables as follows:

$$S(t) - S^* = x(t), \quad I(t) - I^* = y(t), \quad R(t) - R^* = z(t),$$

system (1) becomes

$$\begin{aligned}
\frac{d}{dt}x(t) = {} & -(\beta I^* + \mu_1 - b)x(t) + by(t) + bz(t) \\
& -\beta S^* \int_0^h f(s)y(t-s)ds - \beta x(t)\int_0^h f(s)y(t-s)ds \\
\frac{d}{dt}y(t) = {} & \beta I^* x(t) - (\mu_2 + \lambda)y(t) \\
& +\beta S^* \int_0^h f(s)y(t-s)ds + \beta x(t)\int_0^h f(s)y(t-s)ds \\
\frac{d}{dt}z(t) = {} & \lambda y(t) - \mu_3 z(t).
\end{aligned} \tag{5}$$

Define

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},$$

$$A = \begin{pmatrix} -(\beta I^* + \mu_1 - b) & b - \beta S^* & b \\ \beta I^* & 0 & 0 \\ 0 & \lambda & -\mu_3 \end{pmatrix},$$

$$GX_t = \begin{pmatrix} \beta S^* \int_0^h f(s) \int_{t-s}^t y(u)duds \\ -\beta S^* \int_0^h f(s) \int_{t-s}^t y(u)duds \\ 0 \end{pmatrix},$$

$$F(X_t) = \begin{pmatrix} -\beta x(t) \int_0^h f(s)y(t-s)ds \\ \beta x(t) \int_0^h f(s)y(t-s)ds \\ 0 \end{pmatrix}.$$

We have the following neutral functional differential equation by (5)

$$\frac{d}{dt}(X(t) - GX_t) = AX(t) + F(X_t). \tag{6}$$

Let us first show that A is a stable matrix. In fact, it is easy to find that the characteristic equation of A is

$$\Lambda^3 + a_1\Lambda^2 + a_2\Lambda + a_3 = 0,$$

where

$$a_1 = \beta I^* + \mu_1 - b + \mu_3 > 0,$$

$$a_2 = \mu_3(\beta I^* + \mu_1 - b) + \beta I^*(\mu_2 + \lambda - b) > 0$$

and

$$a_3 = \beta I^* \left(\mu_3(\mu_2 + \lambda) - b(\mu_3 + \lambda)\right) > 0$$

by (3). Furthermore, after a lengthy computation, we can show that

$$\begin{aligned} a_1 a_2 - a_3 &= \frac{\mu_3(b - \mu_1)}{(\mu_3(\mu_2 + \lambda) - b(\mu_3 + \lambda))^2} \{b(b - \mu_1)(\mu_3 + \lambda) \\ &\quad \times [b(\mu_3 + \lambda) + (\mu_2 + \lambda)(\mu_2 + \lambda - b)] \\ &\quad + b\left(\mu_3(\mu_2 + \lambda) - b(\mu_3 + \lambda)\right) \\ &\quad \times [(\mu_2 + \lambda)(\mu_3 + \lambda) + \mu_3(\mu_3 - \mu_2)]\} \\ &> 0. \end{aligned}$$

This shows that A is a stable matrix.

From the stability of matrix A, we can find a positive definite symmetric matrix W such that

$$A^T W + WA = -2E,$$

where E is a unit matrix.

The following inequalities will be used.

Lemma 3. *For any vectors $X, Y \in R^2$ and real matrix $Q = (q_{ij})_{2\times 2}$,*

$$X^T Q X \leq ||X|| ||Q|| ||Y||,$$

where where $||.||$ denotes a Euclidean matrix or vector norm.

Lemma 4 [10]. *For any constants $a > 0$, $b \geq 0$ and $c \geq 0$,*

$$-ac^2 + bc \leq -\frac{1}{2}ac^2 + \frac{b^2}{2a}.$$

The following theorem shows that E_+ is locally asymptotically stable for a sufficiently small delay h.

Theorem 5. (a) *If delay h is small enough such that*

$$h < \min\left\{\frac{1}{\beta S^*}, \frac{1}{\sqrt{2}\beta S^* ||A^T W||}\right\},$$

then the trivial solution of (6) *is locally asymptotically stable.*

(b) *For sufficiently small positive constant δ and delay h such that $h\beta S^* < 1$ and*

$$\sqrt{2}\beta\delta ||W|| + h\beta S^* \left(\sqrt{2}||A^T W|| + 2\beta\delta ||W||\right) < 1, \tag{7}$$

there exists an attractive region $D = D(\delta) \subset C$ for the solutions of (6), *that is, for any $\varphi \in D$, solution $X(t) = (x(t), y(t), z(t))^T$ of* (6) *with the initial function φ satisfies that*

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} z(t) = 0.$$

Here the region D is given explicitly by the parameter values.

Proof. Let us first prove (b).

Define the Liapunov functional

$$\begin{aligned} V(X_t) &= (X(t)-GX_t)^T W (X(t)-GX_t) \\ &\quad + k\int_0^h f(s)\int_{t-s}^t \int_r^t y^2(u)\,du\,dr\,ds, \end{aligned}$$

where k is some positive constant chosen later. For any $X \in R^3$, let us use the notation $||X||$ as a Euclidean norm of X. Thus, it follows from Lemma 3 that the time derivative of $V(X_t)$ along the solutions of (6) becomes for $t \geq t_0$,

$$\begin{aligned} \dot V(X_t)|_{(6)} &= -2||X(t)||^2 - 2X^T(t)A^TW(GX_t) \\ &\quad +2F^T(X_t)WX(t) - 2F^T(X_t)W(GX_t) \\ &\quad +k\int_0^h sf(s)ds\,y^2(t) - k\int_0^h f(s)\int_{t-s}^t y^2(u)\,du\,ds \\ &\leq -2||X(t)||^2 + 2||A^TW||\,||X(t)||\,||GX_t|| \\ &\quad +2||W||\,||X(t)||\,||F(X_t)|| + 2||W||\,||GX_t||\,||F(X_t)|| \\ &\quad +k\int_0^h sf(s)ds\,y^2(t) - k\int_0^h f(s)\int_{t-s}^t y^2(u)\,du\,ds. \end{aligned}$$

Clearly, we have for $t \geq t_0$

$$||GX_t|| = \sqrt{2}\beta S^* \int_0^h f(s)\int_{t-s}^t |y(u)|\,du\,ds.$$

If

$$||y_t|| = \max_{0\leq s\leq h} |y(t-s)| \leq \delta$$

for $t \geq t_0$ and for some positive constant δ, then

$$||F(X_t)|| = \sqrt{2}\beta|x(t)|\int_0^h f(s)|y(t-s)|\,ds \leq \sqrt{2}\beta\delta||X(t)||.$$

Hence, by condition (7), whenever $||y_t|| \leq \delta$ for $t \geq t_0$,we have

$$\begin{aligned} \dot V(X_t)|_{(6)} &\leq \int_0^h f(s)\Big\{-2\left(1-\sqrt{2}\beta\delta||W||\right)||X(t)||^2 \\ &\quad + 2\beta S^*\left(\sqrt{2}||A^TW|| + 2\beta\delta||W||\right)||X(t)||\int_{t-s}^t |y(u)|\,du\Big\}\,ds \\ &\quad +k\int_0^h sf(s)ds\,y^2(t) - k\int_0^h f(s)\int_{t-s}^t y^2(u)\,du\,ds. \qquad (8) \end{aligned}$$

By using Lemma 4, we see that whenever $||y_t|| \leq \delta$ for $t \geq t_0$,

$$\begin{aligned}
\dot{V}(X_t)|_{(6)} &\leq \frac{1}{1-\sqrt{2}\beta\delta||W||}\int_0^h f(s)\left\{-\left(1-\sqrt{2}\beta\delta||W||\right)^2||X(t)||^2\right. \\
&\quad \left. + (\beta S^*)^2\left(\sqrt{2}||A^TW|| + 2\beta\delta||W||\right)^2\left(\int_{t-s}^t |y(u)|du\right)^2\right\} ds \\
&\quad + k\int_0^h sf(s)dsy^2(t) - k\int_0^h f(s)\int_{t-s}^t y^2(u)duds \\
&\leq \frac{1}{1-\sqrt{2}\beta\delta||W||}\left\{-\left(1-\sqrt{2}\beta\delta||W||\right)^2||X(t)||^2\right. \\
&\quad \left. + h(\beta S^*)^2\left(\sqrt{2}||A^TW|| + 2\beta\delta||W||\right)^2\int_0^h f(s)\int_{t-s}^t y^2(u)duds\right\} \\
&\quad + khy^2(t) - k\int_0^h f(s)\int_{t-s}^t y^2(u)duds. \qquad (9)
\end{aligned}$$

We have used Schwartz's inequality in the last inequality of (9). Now let us choose a positive number k as

$$k = \frac{h(\beta S^*)^2\left(\sqrt{2}||A^TW|| + 2\beta\delta||W||\right)^2}{1-\sqrt{2}\beta\delta||W||},$$

which is positive by assumption (7). From (9) and k defined by the above, we have that whenever $||y_t|| \leq \delta$ for $t \geq t_0$,

$$\begin{aligned}
\dot{V}(X_t)|_{(6)} &\leq \left\{-\left[\left(1-\sqrt{2}\beta\delta||W||\right)^2 - (h\beta S^*)^2\left(\sqrt{2}||A^TW|| + 2\beta\delta||W||\right)^2\right]y^2(t)\right. \\
&\quad \left. -\left(1-\sqrt{2}\beta\delta||W||\right)^2\left(x^2(t)+z^2(t)\right)\right\}/(1-\sqrt{2}\beta\delta||W||). \qquad (10)
\end{aligned}$$

Thus, it follows from (7) and (10) that whenever $||y_t|| \leq \delta$ for $t \geq t_0$,

$$\dot{V}(X_t)|_{(6)} \leq -\eta\left(x^2(t)+y^2(t)+z^2(t)\right) \qquad (11)$$

for some positive constant η.

Let us now show that there is a subset $D = D(\delta)$ of C such that for any $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in D$, solution $X(t) = (x(t), y(t), z(t))^T$ of (6) through (t_0, φ) must satisfy $||y_t|| \leq \delta$ for $t \geq t_0$.

In fact, we can choose D as follows:

$$\begin{aligned}
D = \{\varphi \in C \ \mid \ &||\varphi(0) - G\varphi|| < \delta(1-\beta S^*h), \\
&V(\varphi) < L, \ \ ||\varphi|| \leq \delta(1-\beta S^*h)\}, \qquad (12)
\end{aligned}$$

where L is defined as

$$\begin{aligned} L &= \inf_{\|\varphi(0)-G\varphi\|=\delta(1-\beta S^* h)} V(\varphi) \\ &\geq \inf_{\|\varphi(0)-G\varphi\|=\delta(1-\beta S^* h)} \left\{(\varphi(0)-G\varphi)^T W(\varphi(0)-G\varphi)\right\} > 0, \end{aligned}$$

since $1 > \beta S^* h$ and W is positive definite.

Let us first show that $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in D$ implies that for $t \geq t_0$,

$$\|X(t) - GX_t\| \leq \delta(1 - \beta S^* h). \tag{13}$$

If not, there is some $\bar{t} > t_0$ such that (13) holds for $t_0 \leq t \leq \bar{t}$, and $\|X(\bar{t}) - GX_{\bar{t}}\| = \delta(1 - \beta S^* h)$. Thus, $V(X_{\bar{t}}) \geq L$.

On the other hand, it follows from (13) that for $t_0 \leq t \leq \bar{t}$,

$$\begin{aligned} |y(t)| &\leq \delta(1-\beta S^* h) + \beta S^* \int_0^h f(s) \int_{t-s}^t |y(u)| du ds \\ &\leq \delta(1-\beta S^* h) + \beta S^* h \max_{0 \leq s \leq h} |y(t-s)| \\ &\leq \delta(1-\beta S^* h) + \beta S^* h \max_{t_0 - h \leq s \leq t} |y(s)|. \end{aligned}$$

Thus, for $t_0 \leq t \leq \bar{t}$,

$$\max_{t_0-h \leq s \leq t} |y(s)| \leq \delta(1-\beta S^* h) + \beta S^* h \max_{t_0-h \leq s \leq t} |y(s)|,$$

from which we have that for $t_0 \leq t \leq \bar{t}$,

$$\|y_t\| \leq \max_{t_0-h \leq s \leq t} |y(s)| \leq \delta. \tag{14}$$

Therefore, it follows from (8) that

$$V(X_{\bar{t}}) < V(\varphi) < L,$$

which contradicts to $V(X_{\bar{t}}) \geq L$. This proves that (13) holds for $t \geq t_0$.

By the same argument as used in (14) we can show that $\|y_t\| \leq \delta$ for $t \geq t_0$. From (11) we have that

$$\int_{t_0}^{+\infty} \left(x^2(t) + y^2(t) + z^2(t)\right) dt < +\infty.$$

Let us further show that for any $\varphi \in D$, the solution $(x(t), y(t), z(t))^T$ of (6) through (t_0, φ) is bounded.

In fact, it is easy to see that there are two positive constants M_1 and M_2 ($M_1 \geq M_2$) which are independent of φ such that for $t \geq t_0$,

$$M_2^2||X(t) - GX_t||^2 \leq V(X_t) < V(\varphi) \leq M_1^2||\varphi||^2.$$

Thus, we have that for $t \geq t_0$,

$$\begin{aligned} |x(t)| &\leq h\beta S^* \max_{0\leq s\leq h} |y(t-s)| + \frac{M_1}{M_2}||\varphi|| \\ &\leq h\beta S^* \max_{t_0-h\leq s\leq t} |y(s)| + \frac{M_1}{M_2}||\varphi||, \end{aligned} \tag{15}$$

$$\begin{aligned} |y(t)| &\leq h\beta S^* \max_{0\leq s\leq h} |y(t-s)| + \frac{M_1}{M_2}||\varphi|| \\ &\leq h\beta S^* \max_{t_0-h\leq s\leq t} |y(s)| + \frac{M_1}{M_2}||\varphi||, \end{aligned} \tag{16}$$

and

$$|z(t)| \leq \frac{M_1}{M_2}||\varphi||. \tag{17}$$

Clearly, (15) and (16) imply that for $t \geq t_0$,

$$|y(t)| \leq \max_{t_0-h\leq s\leq t} |y(s)| \leq \frac{M_1}{M_2(1-\beta S^* h)}||\varphi||,$$

$$|x(t)| \leq \frac{M_1}{M_2(1-\beta S^* h)}||\varphi||,$$

which together with (17) shows boundedness of $(x(t), y(t), z(t))^T$.

Note that from (6) , we see that $\frac{d}{dt}(x^2(t) + y^2(t) + z^2(t))$ is also bounded for $t \geq t_0$. By the well-known Barbălat's lemma [2], we have that $\lim_{t\to+\infty}(x^2(t) + y^2(t) + z^2(t)) = 0$. This proves (b).

Conclusion (a) immediately follows from (7), (15), (16) and (17) as long as we choose δ sufficiently small. The proof of Theorem 5 is completed.

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